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Norm Estimates for Inverses of Euclidean Distance Matrices

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In this paper, we obtain a better estimate for the norm of inverses of Euclidean distance matrices of low dimensions. © 1992 Academic Press, Inc.

1. INTRODUCTION

We work in the Euclidean space $\mathbb{R}^d$, the dimension d being fixed. Let x_1, $x_2, ..., x_n$ be n distinct points (called nodes) in R^d, and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$. Schoenberg [8] proved that the $n \times n$ distance matrix $A = (\|x_j - x_k\|)$ has exactly 1 positive eigenvalue and $(n-1)$ negative eigenvalues. As a consequence of Schoenberg's result, the following interpolation problem is soluble: Given arbitrary data $\{b_1, b_2, ..., b_n\}$ on the node set $\{x_1, x_2, ..., x_n\}$, find a unique function f in the linear span of the n functions $\|x - x\|$, $\|x - x_2\|$, ..., $\|x - x_n\|$, such that

$$f(x_j) = b_j, \qquad (1 \leqslant j \leqslant n).$$

This interpolation method is a natural generalization of the piecewise linear interpolation on the real line, and is an important special case of the radial basis function interpolation. See the review papers by Dyn [3] and Powell [6].

In implementing the interpolation scheme, it is important to have an estimate for the norm of A^{-1}. Here we look at A^{-1} as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$, and use the matrix norm subordinate to the Euclidean norm on $\mathbb{R}^n$. We also denote the matrix norm by $\|\cdot\|$, as no confusion is likely to occur. Let λ_1, $\lambda_2, ..., \lambda_n$ be all the eigenvalues of A, and let $\lambda = \min\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\}$. Since A is a real and symmetric matrix, it is elementary to see $\|A^{-1}\| = 1/\lambda$.

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Ball [2] recently proved the following interesting result:

THEOREM 1 (Ball). *Let $x_1, x_2, ..., x_n$ be n points in $\mathbb{R}^d$, where d is an odd integer. If $\|x_j - x_k\| \geqslant \varepsilon$ for al $j \neq k$, then all the eigenvalues of A have absolute values at least*

$$\varepsilon 2^{d-1} \binom{d-1}{\frac{d-1}{2}}^{-1} \gamma_d, \tag{1}$$

where γ_d is the distance in $C[-1, 1]$ of the function $|x|$ from the space of polynomials of degree $(d-1)$ or less.

Ball [2] asserted that the estimate (1) is best possible for the case $d=1$ but is not best possible for the case $d=3$. Ball also conjectured that the estimate is not best possible for $d=5, 7, 9, ...$.

An estimate was given by Narcowich and Wartd [5] for the more general matrix

$$A_\alpha = (\|x_j - x_k\|^\alpha), \qquad 0 < \alpha < 2.$$

Nevertheless, when $\alpha = 1$, and d ($\geqslant 3$) is an odd integer, their estimate is not as sharp as the one given by Ball.

Ball [2] pointed out that it is an interesting geometric problem to determine the best possible constant, at least for $d=2$ and 3. In this paper, we provide an estimate for all dimensions d. Asymptotically, our estimate is weaker than Ball's; however, it yields better results for low dimensions. And our estimate is best possible for $d=1$.

2. THE ESTIMATE

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of the matrix $A = (\|x_j - x_k\|)$ in descending order. By Schoenberg's result mentioned in Section 1, the following inequalities are true:

$$\lambda_1 > 0 > \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

Since the trace of A is 0, we have $\sum_{j=1}^n \lambda_j = 0$. Hence $\lambda_1 = \sum_{j=2}^n |\lambda_j|$. Thus λ_2 is one of the eigenvalues having the smallest absolute value. By the Courant Fischer Theorem,

$$\lambda_2 = \min_{\dim V = n-1} \max_{v \in V, \|v\| = 1} v^T A v \leqslant \max_{v^T u = 0, \|v\| = 1} v^T A v,$$

where u is the vector $(1, 1, ..., 1)^T$. Formula (5) allows us to use Fourier transform techniques to estimate the quadratic form $v^T A v$ for $v^T u = 0$. A similar method has been used by Narcowich and Ward [4].

DEFINITION 2. The Fourier transform of a function f *in* $L^1(\mathbb{R}^d)$ *is the function* $\hat{f}$ *defined by*

$$\hat{f}(t) = (2\pi)^{-d/2} \int_{R^d} e^{-ixt} f(x)\, dx.$$

The inverse Fourier transform of f is the function $\check{f}$ *defined by*

$$\check{f}(t) = (2\pi)^{-d/2} \int_{R^d} e^{ixt} f(x)\, dx.$$

We also use the symbol $\mathscr{F}(f)$ to denote the Fourier transform of f, and the symbol $\mathscr{F}^{-1}(f)$ to denote the inverse Fourier transform of f. It is well known that if both f and $\mathscr{F}(f)$ belong to $L^1(\mathbb{R}^d)$ then $f = \mathscr{F}^{-1}(\mathscr{F}(f))$; see Rudin [7].

LEMMA 3. *Let*

$$B_1(x) = \begin{cases} 1, & \text{if } \|x\| \leqslant 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\hat{B}_1$ *is radial and*

$$\hat{B}_1(x) = (2r)^{-d/2} J_{d/2}(r/2), \qquad r = \|x\|,$$

where J_ν *is the Bessel function of the first kind.*

Proof. The Lemma is trivial in the case $d = 1$. So we assume that $d \geqslant 2$. Since B_1 is radial, so is $\hat{B}_1$ [9, p. 135]. By Theorem 3.3 in [9, p. 155], we have

$$\hat{B}_1(x) = r^{-(d-2)/2} \int_0^{1/2} r^{d/2} J_{(d-2)/2}(r\tau)\, d\tau, \qquad r = \|x\|.$$

A change of variable $\rho = r\tau$ in this integral leads to

$$\hat{B}_1(x) = r^{-d} \int_0^{r/2} \rho^{d/2} J_{(d-2)/2}(\rho)\, d\rho, \qquad r = \|x\|. \tag{2}$$

The following formula for Bessel functions can be found in Watson [10, p. 45]

$$\frac{d}{dz}\{z^{\nu}J_{\nu}(z)\}=z^{\nu}J_{\nu-1}(z).$$

Hence

$$z^{\nu}J_{\nu}(z)\,|_a^b=\int_a^b z^{\nu}J_{\nu-1}(z)\,dz. \tag{3}$$

Applying Eq. (3) to the integral in Eq. (2) with $\nu=d/2$, $a=0$, $b=r/2$, we get

$$\int_0^{r/2}\rho^{d/2}J_{(d-2)/2}(\rho)\,d\rho=(r/2)^{d/2}\,J_{d/2}(r/2).$$

It follows that

$$\hat{B}_1(r)=(2r)^{-d/2}\,J_{d/2}(r/2). \quad \blacksquare$$

LEMMA 4. *Let B_1 be as in Lemma 3, and let*

$$B_2(x)=(B_1*B_1)(x)=(2\pi)^{-d/2}\int_{\|y\|\leqslant 1/2}B_1(x-y)\,dy.$$

Then the following results are true:

1. $B_2(0)=2^{(2-3d)/2}/d\Gamma(d/2)$.
2. $\operatorname{supp}(B_2)=\{x:\|x\|\leqslant 1\}$.
3. $\hat{B}_2(x)=(2\,\|x\|\,)^{-d}\,J^2_{d/2}(\|x\|/2)$. *Consequently*, $\hat{B}_2\in L^1(\mathbb{R}^d)$.

Proof. To prove Part 1, using polar coordinates, we write

$$\begin{aligned}B_2(0)&=(2\pi)^{-d/2}\int_{\|y\|\leqslant 1/2}B_1(-y)\,dy=(2\pi)^{-d/2}\int_{\|y\|\leqslant 1/2}1\,dy\\&=(2\pi)^{-d/2}\cdot\frac{2\pi^{d/2}}{d2^d\Gamma(d/2)}=\frac{2^{(2-3d)/2}}{d\Gamma(d/2)}.\end{aligned}$$

Part 2 is obvious.

To prove Part 3, we recall that the Fourier transform maps the convolution $B_1 * B_1$ to the product $\hat{B}_1 \cdot \hat{B}_1$; see Rudin [7, Theorem 7.2, p. 167]. By Lemma 3 and the definition of B_2, we have $\hat{B}_2(x) = (2\,\|x\|)^{-d} J^2_{d/2}(\|x\|/2)$. We observe that $\hat{B}_2$ is finite at the origin and that

$$|\hat{B}_2(x)| \leqslant (1+\delta)\,\frac{4}{\pi}\,2^d\,\|x\|^{-(d+1)}$$

for $\|x\|$ large, where δ is a positive constant; see Stein and Weiss [9, p. 158]. It follows that $\hat{B}_2 \in L^1(\mathbb{R}^d)$. ∎

We remark here that since B_2 is radial we have $\mathscr{F}(B_2) = \mathscr{F}^{-1}(B_2)$. Therefore $\mathscr{F}(\hat{B}_2) = \mathscr{F}^{-1}(\hat{B}_2) = B_2$.

Let S_{d-1} denote the unit sphere in $\mathbb{R}^d$. Let Ω_d denote the Fourier transform of the rotational invariant probability measure on S_{d-1}, that is,

$$\Omega_d(x) = w_{d-1}^{-1} \int_{S_{d-1}} e^{ixw}\,dw,$$

where dw denotes the usual measure on S_{d-1} and w_{d-1} the area of S_{d-1}. Ω_d is radial and can be expressed in terms of the Bessel function; see [8],

$$\Omega_d(t) = \Gamma\left(\frac{d}{2}\right)\left(\frac{2}{t}\right)^{d/2-1} J_{d/2-1}(t).$$

The following lemma concerns the integral representation of the function $\|x\|^\alpha$ $(0<\alpha<2)$ by the function Ω_d.

LEMMA 5. *Let $0<\alpha<2$. Then the following identity is true*:

$$\|x\|^\alpha = \frac{2^{1+\alpha}\Gamma((\alpha+d)/2)}{\Gamma(-\alpha/2)\,\Gamma(d/2)} \int_0^\infty r^{-(1+\alpha)}[\Omega_d(r\,\|x\|)-1]\,dr.$$

Proof. Observe that Ω_d is real, bounded in absolute value by 1, and the function $\Omega_d(r)-1$ has a zero of order 2 at the origin, so the integrand in the above representation is absolutely integrable. Hence, using dilation invariance, we see that there exists a constant c, such that

$$\|x\|^\alpha = c \int_0^\infty r^{-(1+\alpha)}[\Omega_d(r\,\|x\|)-1]\,dr.$$

What remains in the proof is to verify that the constant is correct. To do this, integrate both sides against the function $(2\pi)^{-d/2}\, e^{-\|x\|^2/2}$ over $\mathbb{R}^d$. On the left hand side, we have

$$(2\pi)^{-d/2} \int_{\mathbb{R}_d} \|x\|^\alpha\, e^{-\|x\|^2/2}\, dx = \frac{w_{d-1}}{(2\pi)^{d/2}} \int_0^\infty t^{\alpha+d-1} e^{-t^2/2}\, dt$$
$$= \frac{2^{\alpha/2}\Gamma((\alpha+d)/2)}{\Gamma(d/2)}.$$

On the right hand side, we have

$$(2\pi)^{-d/2} \int_{\mathbb{R}_d} e^{-\|x\|^2/2} \left\{ c \int_0^\infty r^{-(1+\alpha)} [\Omega_d(r\, \|x\|) - 1]\, dr \right\} dx$$
$$= c w_{d-1}^{-1} \int_0^\infty r^{-(1+\alpha)} \left\{ \int_{S_{d-1}} dw \left[(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{irxw} e^{-\|x\|^2/2}\, dx \right] - 1 \right\} dr$$
$$= c \int_0^\infty r^{-(1+\alpha)} (e^{-r^2/2} - 1)\, dr$$
$$= c \int_0^\infty (e^{-r^2/2} - 1)\, d\left(-\frac{r^{-\alpha}}{\alpha} \right)$$
$$= -\frac{c}{\alpha} \int_0^\infty r^{1-\alpha} e^{-r^2/2}\, dr$$
$$= -\frac{c\Gamma(1-\alpha/2)}{\alpha 2^{\alpha/2}}.$$

Here we used the fact that the function $e^{-\|x\|^2/2}$ is invariant under the Fourier transform. Thus we have

$$c = \frac{2^{1+\alpha}\Gamma((\alpha+d)/2)}{\Gamma(-\alpha/2)\,\Gamma(d/2)}. \quad \blacksquare$$

THEOREM 6. *Let $x_1, x_2, ..., x_n \in \mathbb{R}^d$ with $\min_{j \neq k} \|x_j - x_k\| = \varepsilon > 0$. Then all the eigenvalues of the matrix $A = (\|x_j - x_k\|)$ have absolute values at least*

$$\frac{\varepsilon\Gamma((d+1)/2)}{\sqrt{\pi}\, d\Lambda_d \Gamma(d/2)},$$

where $\Lambda_d = \sup_{r \geqslant 0} [r J_{d/2}^2(r)]$.

Proof. We first explain that we may assume $\varepsilon = 1$ without loss of generality. Indeed, let $x_j' = \varepsilon^{-1} x_j$. We then have

$$\min_{j \neq k} \|x_j' - x_k'\| = \min_{j \neq k} \varepsilon^{-1}\, \|x_j - x_k\| = 1.$$

If $\lambda_1, ..., \lambda_n$ are the eigenvalues of the matrix $A' = (\|x'_j - x'_k\|)$, then $\varepsilon\lambda_1, ..., \varepsilon\lambda_n$ are the eigenvalues of the matrix $A = (\|x_j - x_k\|)$.

Using Lemma 5 with $\alpha = 1$, we have

$$\|x\| = \Delta_d \int_0^\infty r^{-2}[\Omega_d(r\,\|x\|) - 1]\,dr,$$

where $\Delta_d := -2\Gamma((d+1)/2)/\Gamma(1/2)$.

Let $v \in \mathbb{R}^n$, $v^T u = 0$. We have

$$\begin{aligned} v^T A v &= \Delta_d \int_0^\infty r^{-2} \sum_{j,k=1}^n v_j v_k \Omega_d(r\,\|x_j - x_k\|)\,dr \\ &= \Delta_d w_{d-1}^{-1} \int_0^\infty r^{-2} \left[\int_{S_{d-1}} \sum_{j,k=1}^n v_j v_k e^{ir(x_j - x_k)w}\,dw\right] dr \\ &= \Delta_d w_{d-1}^{-1} \int_0^\infty r^{-2} \left[\int_{S_{d-1}} \Big|\sum_{j=1}^n v_j e^{ix_j rw}\Big|^2\,dw\right] dr \\ &= \Delta_d w_{d-1}^{-1} \int_{\mathbb{R}^d} \Big|\sum_{j=1}^n v_j e^{ix_j rw}\Big|^2 \|t\|^{-(d+1)}\,dt. \end{aligned}$$

For all $r \geqslant 0$, by the definition of the number Λ_d, we have $rJ_{d/2}^2(r) \leqslant \Lambda_d$. Consequently, $rJ_{d/2}^2(r/2) \leqslant 2\Lambda_d$. Hence

$$\frac{rJ_{d/2}^2(r/2)}{r^{d+1}} \leqslant \frac{2\Lambda_d}{r^{d+1}} \qquad (r > 0).$$

Since Δ_d is a negative constant, it follows that

$$\begin{aligned} v^T A v &= \Delta_d (2w_{d-1}\Lambda_d)^{-1} \int_{\mathbb{R}^d} \left|\sum_{j=1}^n v_j e^{ix_j t}\right|^2 \frac{2\Lambda_d}{\|t\|^{d+1}}\,dt \\ &\leqslant \Delta_d (2w_{d-1}\Lambda_d)^{-1} \int_{\mathbb{R}^d} \left|\sum_{j=1}^n v_j e^{ix_j t}\right|^2 \frac{\|t\|\, J_{d/2}^2(\|t\|/2)}{\|t\|^{d+k}}\,dt \\ &= 2^d \Delta_d (2w_{d-1}\Lambda_d)^{-1} \int_{\mathbb{R}^d} \left[\sum_{j,k=1}^n v_j v_k e^{i(x_j - x_k)t}\right] \hat{B}_2(\|t\|)\,dt, \end{aligned}$$

where the function B_2 is defined as in Lemma 4.

By Lemma 4 and the remark following it, we have

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x_j - x_k)t}\, \hat{B}_2(\|t\|)\,dt = B_2(\|x_j - x_k\|).$$

Therefore

$$v^T A v \leqslant 2^d (2\pi)^{d/2} \Delta_d (2w_{d-1} A_d)^{-1} \sum_{j,k=1}^{n} v_j v_k B_2(\|x_j - x_k\|).$$

Finally, because $\min_{j \neq k} \|x_j - x_k\| = 1$, and $\sup(B_2) = \{x : \|x\| \leqslant 1\}$, we obtain

$$v^T A v \leqslant [2^d (2\pi)^{d/2} \Delta_d (2w_{d-1} A_d)^{-1} B_2(0)] \sum_{j=1}^{n} v_j^2.$$

$$= -\frac{\Gamma((d+1)/2)}{\sqrt{\pi}\, d A_d \Gamma(d/2)} \sum_{j=1}^{n} v_j^2.$$

According to the remarks at the beginning of this section, we see that the proof is completed by an application of the Courant–Fischer Theorem. ∎

The estimate given by Theorem 6 holds for all dimension d. When $d = 1$, we have (see [10, p. 54])

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z.$$

Thus $A_1 = 2/\pi$ and estimate (3) gives $\varepsilon/2$. The author was informed by B. J. C. Baxter that he and M. Powell had verified that the estimate $\varepsilon/2$ is best possible for the case $d = 1$. When $d = 3$, we have (see [10, p. 54])

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right).$$

Numerical experiment suggests that $A_3 < 2.4/\pi$, and estimate (3) gives a bound which is greater than $\varepsilon/3.6$, and which is better than the one given by Theorem 1 where the bound is given to be $\varepsilon/4$. And this confirms Ball's assertion that $\varepsilon/4$ is not best possible. However, according to Formula 9.3.5 in [1], $J_m(m)$ is of the order $m^{-1/3}$ so that our estimate is of the order $\varepsilon d^{-5/6}$ asymptotically, thus, our estimate is weaker than the one given by Theorem 1 for d sufficiently large.

It would be interesting to determine if the estimate given by Theorem 6 is best possible for $d = 2, 3$. The problem seems to be related to sphere packings in $\mathbb{R}^d$. We caution that the sphere packing problem in $\mathbb{R}^d$ has not been settled for $d \geqslant 3$.

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